Consequently, the equilibrium position is asymptotically stable relative to $q_1, \ldots, q_m, q_1^{\cdot}, \ldots, q_n$, T_2 uniformly with respect to initial conditions from domain (3.6).

Note . It is easy to show analogously [5] that in the case being considered here the dissipative forces possess partial dissipation or are entirely absent ($f \equiv 0$), while in the set $H > 0$ there are no motions of the whole system as a single solid body (see Zhukovskii's theorem in [4] p. 67), then in the perturbed motion $H \to 0$ as $t \to \infty$ and, what is more, uniformly in domain (3.6), which is proved analogously to the above. Consequently, the conclusion on asymptotic stability relative to $q_1, \ldots, q_m, q_1^{\cdot}, \ldots, q_n^{\cdot}$, T_2, uniform with respect to initial conditions from domain (3.6), remains in force. From this, in the special case when the potential energy U has a minimum at the equilibrium position, there follows an addition to Theorem 1.1 of [5] concerning uniformity with respect to initial conditions from domain (3.6).

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APPROXIMATE SYNTHESIS METHOD FOR OPTIMAL CONTROL OF A SYSTEM SUBJECTED TO RANDOM PERTURBATIONS

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An approximate method is proposed for synthesizing the optimal control for a dynamical system in the presence of external random perturbations and measurement errors. The synthesis problem posed reduces, as is known, to solving a nonlinear parabolic partial differential equation (the Bellman equation) whose exact solutions are known only in a few cases. It is assumed that either the external perturbations acting on the system are sufficiently small or the measurement errors are large. Under these conditions the Bellman equation contains a small parameter in the leading derivative, and the solution is constructed by means of an expansion with respect to the small parameter. It is shown that the approximate synthesis of the optimal control for perturbed systems can be constructed in explicit form if the solution of the corresponding problem for the perturbation-free system is known. Estimates of the errors in the method are proved. Examples are given.

1. Statement of the problem. We consider the problem of the optimal control of the final state of a system whose motion is described by the equation

$$dx(t)/dt = A(t)x(t) + b(t, u) + C(t)\xi(t) \tag{1.1}$$

Here t is time, x is the n-dimensional phase coordinate vector, u is the m-dimensional control vector, $b(t, u)$ is a given vector-valued function, $\xi(t)$ is the s-dimensional vector-valued function of random perturbations acting on the system. The matrices $A(t)$, $C(t)$ are given time functions and have the dimension $n \times n$, $n \times s$, respectively. It is assumed at each instant t the vector $\xi(t)$ is normally distributed with zero mean and covariance matrix $G(t)$ and is uncorrelated in time, i.e. is a white noise. The intensity of the external perturbations is assumed small, i.e. $G = \varepsilon G_*$, where ε is a small parameter $(0 \leqslant \varepsilon \lll 1)$ and $G_*(t)$ is a matrix with bounded elements. The constraint

$$u(t) \in U \tag{1.2}$$

where U is a specified bounded closed set, is imposed on the control vector. It is assumed that an exact measurement of the system's phase vector $x(t)$ is possible at any instant t. We are required to find a control u, depending on time and on the instantaneous phase vector $x(t)$, which under condition (1.2) minimizes the mean of a scalar function $F(x(T))$ of the phase coordinates at the final instant T.

We introduce the Bellman function $S(t, x)$, equal to the optimal value of the functional to be minimized, under the condition that the process starts at instant t with the phase vector $x(t) = x$. Then, as is well known [1], the optimal control problem posed reduces to the solving of the nonlinear parabolic equation

$$S_t - H(S_x, x, t) + {}^1/_2 \varepsilon \operatorname{Sp}(LS_{xx}) = 0 \tag{1.3}$$

with the initial condition

$$S(T, x) = F(x) \tag{1.4}$$

Here S_t is the partial derivative with respect to time, S_x is the vector of first partial derivatives, S_{xx} is the matrix of second partial derivatives of function S with respect to the components of vector x, Sp denotes the trace of the matrix, and, furthermore, we have introduced the notation

$$H(S_x, x, t) = -\min_{u \in U}(S_x, Ax + b(t, u)) \qquad L(t) = CG_*C' \tag{1.5}$$

where the parantheses denote the scalar product of vectors.

For the sake of generality we assume that the process termination instant T is not fixed and is determined by the condition $h(T, x(T)) = 0$, and that the mean of the function $F(T, x(T))$ is to be minimized at the end of the process. Here $h(t, x)$, $F(t, x)$ are given functions and in the case of fixed T we have $h = T - t$. Under the assumptions made the function S satisfies the condition

$$S(t, x) = F(t, x) \quad \text{for} \quad h(t, x) = 0 \tag{1.6}$$

instead of (1.4). To be definite we take $h > 0$ at the start of the process. The solution of the Cauchy problem (1.3), (1.6) must be constructed in the region $h \geqslant 0$. Thus, when the external perturbations have small intensity, the optimal control synthesis problem is reduced to solving a Cauchy problem for the nonlinear parabolic equation (1.3) with a small parameter in the leading derivative and with initial condition (1.4) or (1.6). The optimal control is determined after the function S has been found from the condition that a minimum has been achieved in (1.5). We note that the optimal control problem for the system (1.1), (1.2) in the presence of large measurement errors also leads to exactly this same mathematical problem.

Let the measurement process be given by the equation

$$y(t) = Q(t)\, x(t) + \eta(t) \tag{1.7}$$

Here y is the l-dimensional vector of measurement results, $\eta(t)$ is the vector-valued function of measurement errors, which, just as $\xi(t)$, is a white noise with covariance matrix $B(t)$. Suppose that at the initial instant t_0 the random vector $x(t_0)$ is normally distributed with mean x_0 and covariance matrix D_0. Then when the measurement results are processed by the maximum likelihood method, because of linearity of Eqs. (1.1), (1.2) the a posteriori distribution of vector $x(t)$ is normal, while the equations describing the variations of the mean vector $z(t)$ and of the covariance matrix $D(t)$ of this distribution have the following form [2]:

$$\begin{aligned} dz / dt &= Az + b + DQ'B^{-1}(y - Qz), \qquad z(t_0) = x_0 \\ dD / dt &= AD + DA' - DQ'B^{-1}QD + CGC', \quad D(t_0) = D_0 \end{aligned} \tag{1.8}$$

Here the explicit dependency of the functions on time has been omitted, the prime denotes transposition, and the minus one power denotes the inverse matrix. Since (1.8) for D does not depend upon $b(t, u)$, the matrix $D(t)$ can be determined in advance without dependence on the measurement results and on the control law. From now on we take $D(t)$ as a known time function. Once again we consider the problem of determining the optimal control as a function of time and of the given measurements, minimizing the mean of the function $F(x(T))$ at the final instant T. We introduce the Bellman function $S(t, z)$, equal to the optimal value of the functional to be minimized, under the condition that the process starts at instant t and that $z(t) = z_0$ is known at this instant. Then the optimal control problem posed above reduces to the solving of a Cauchy problem for the Bellman equation [1]

$$\begin{gathered} S_t + \min_{u \in U}(S_z, Az + b) + {}^1/_2 \operatorname{Sp}(DQ'B^{-1}QDS_{zz}) = 0 \\ S(T, z) = [(2\pi)^n \det D(T)]^{-1/2} \int_{-\infty}^{+\infty} F(x) \exp\left[-\frac{1}{2}(D^{-1}(T)(z - x), (z - x))\right] dx \end{gathered} \tag{1.9}$$

If the measurement errors are large, i.e. $B^{-1} = \varepsilon B_*$, where B_* is a matrix with bounded elements, ε is a small parameter, then problem (1.9) is equivalent to problem (1.3), (1.4).

2. The small parameter method. We proceed to construct the approximate solution of problem (1.3), (1.6) by the small parameter method. We assume that the functions H, L, F and S are sufficiently smooth and we seek the solution of problem (1.3), (1.6) in the form of a regular expansion in powers of the small parameter

$$S(t, x) = S^\circ(t, x) + \varepsilon S^1(t, x) + \ldots \tag{2.1}$$

and we represent the function H from (1.5) in the form

$$H(S_x, x, t) = H(S_x^\circ, x, t) + \varepsilon(\nabla H(S_x^\circ, x, t), S_x^1(t, x)) + \ldots \tag{2.2}$$

Here and further ∇H is the vector of partial derivatives of function H with respect to the components of vector S_x. We substitute expansions (2.1), (2.2) into (1.3), (1.6) and, restricting ourselves to first-order terms in ε, we find the equations and the initial values for the functions S° and S^1

$$S_t^\circ - H(S_x^\circ, x, t) = 0, \quad S^\circ(t, x) = F(t, x) \quad \text{for} \quad h(t, x) = 0 \tag{2.3}$$

$$S_t^1 - (\nabla H(S_x^\circ, x, t), S_x^1) + {}^1/_2 \operatorname{Sp}(LS_{xx}^\circ) = 0$$
$$S^1(t, x) = 0 \quad \text{for} \quad h(t, x) = 0 \tag{2.4}$$

Analogous equations can be written down also for the higher approximations.

From relations (1.5), (2.3) it follows that H is the Hamiltonian function while $S^\circ(t, x)$ is the Bellman function for the optimal control problem in the absence of random perturbations. We assume that this problem has been solved, i.e. that we have found the synthesis of the optimal control $u^\circ(t, x)$, the function $S^\circ(t, x)$ for the deterministic system, and the corresponding field of optimal trajectories

$$x = \varphi(t, a) \tag{2.5}$$

where $\varphi(t, a)$ is a vector-valued function, a is an n-dimensional vector of arbitrary constants. We assume that equality (2.5) can be solved with respect to a in some region of the phase space and that we can obtain the dependency

$$a = \psi(t, x) \tag{2.6}$$

The first-approximation equation (2.4) is a first-order linear inhomogeneous partial differential equation. The system of equations determining its characteristics has the form

$$dx / dt = -\nabla H(S_x^\circ, x, t), \qquad dS^1 / dt = -{}^1/_2 \operatorname{Sp}(LS_{xx}^\circ) \tag{2.7}$$

Taking into account notation (1.5) and the known equality $S_x^\circ = -p$, where p is the adjoint variable vector for the deterministic system, we note that the first equation in (2.7) determines, according to the maximum principle, the optimal trajectories of the deterministic system. The general solution of this equation is given by (2.5), while the system of first integrals is given by (2.6). Then, form (2.7), (2.5), (2.6) it follows that the solution of the Cauchy problem (2.4) for the function S^1 is expressed as

$$S^1(t, x) = \frac{1}{2} \int_t^{T(x)} \operatorname{Sp}[L(\tau) S_{\zeta\zeta}^\circ(\tau, \zeta)] \, d\tau \tag{2.8}$$

Here $T(x)$ is a root of the equation $h(T, x) = 0$, and the matrix $S_{\zeta\zeta}^\circ$ should be taken with

$$\zeta = \varphi(\tau, \psi(t, x)) \tag{2.9}$$

The synthesis of the optimal control in the zeroth and first approximations, u° and u^1, are obtained from the conditions

$$(S_x^\circ, b(t, u^\circ(t, x))) = \min_{u\in U}(S_x^\circ, b(t, u)) \tag{2.10}$$
$$(S_x^\circ + \varepsilon S_x^1, b(t, u^1(t, x))) = \min_{u\in U}(S_x^\circ + \varepsilon S_x^1, b(t, u))$$

Equalities (2.1), (2.8) – (2.10) determine in explicit form the approximate solution of the Bellman equation and the synthesis of the optimal control for the problem with random perturbations if the solution of the synthesis problem for the perturbation-free system is known. The solution constructed is invalid where the transformation (2.5) is noninvertible. We note further that close to the surface of discontinuity of the Bellman function S° or of its derivatives it is necessary to add to the above-constructed solution, which is regular in ε, a nonregular part of the solution of the boundary layer type.

3. Error estimates for the approximate solution. We examine the errors estimates for the approximate solution of the Cauchy problem (1.3), (1.4) for the case when the process termination instant T is fixed and the mean of the function $F(x(T))$ is to be minimized at the end of the process. In this case the equations and the initial conditions for the Bellman function, as well as the zeroth and first approximation Eqs. (2.3), (2.4), have the form

$$S_t + (Ax, S_x) + M(t, S_x) + {}^1/_2\varepsilon\,\mathrm{Sp}\,(LS_{xx}) = 0, \quad S(T, x) = F(x) \tag{3.1}$$
$$S_t^\circ + (Ax, S_x^\circ) + M(t, S_x^\circ) = 0, \quad S^\circ(T, x) = F(x) \tag{3.2}$$
$$S_t^1 + (Ax, S_x^1) + (\nabla M(t, S_x^\circ), S_x^1) + {}^1/_2\,\mathrm{Sp}\,(LS_{xx}^\circ) = 0, \quad S^1(T, x) = 0 \tag{3.3}$$

Here and further, for the function M, the vector of its first derivatives, and the matrix of second derivatives, we have introduced the notation

$$M(t, S_x) = \min_{u\in U}(S_x, b(t, u)), \quad \nabla M = \partial M / \partial S_x \tag{3.4}$$
$$N = \partial^2 M / dS_x^2$$

We assume that for all $t \in [t_0, T]$, $u \in U$ and $x \in R_n$, where R_n is an n-dimensional Euclidean space with norm $\|x\| = (x_1^2 + \ldots + x_n^2)^{1/2}$, the following conditions are fulfilled:

1. Matrix L is positive definite.
2. The components of vector b and the elements of matrices A and L are bounded in absolute value.
3. Function F is continuous and bounded in absolute value together with its derivatives up to third order inclusive.
4. Function M from (3.4) is continuous and bounded in absolute value together with its derivatives with respect to S_x up to second order inclusive.
5. The solution of problems (3.1) – (3.3) exist and are unique, continuous and bounded in absolute value together with their derivatives up to third order inclusive.

The subsequent estimates are based on the following lemma, directly ensuing (after the substitution $t' = T - t$) from Theorem 10 in [3].

L e m m a. Let the continuous and bounded function $v(t, x)$ serve as the solution of the following Cauchy problem:

$$\sum_{i,j=1}^n \alpha_{ij}(t, x)\frac{\partial^2 v}{\partial x_i \partial x_j} + \sum_{i=1}^n \beta_i(t, x)\frac{\partial v}{\partial x_i} + \gamma(t, x)v + \frac{\partial v}{\partial t} = \delta(t, x)$$
$$v(T, x) = v_0(x)$$

where for all $t \in [t_0, T]$, $x \in R_n$ the matrix α_{ij} is positive definite, we have $|v_0(x)| \leqslant k_1$, $|\delta(t, x)| \leqslant k_2$, $|\gamma(t, x)| \leqslant k_3$, and the coefficients α_{ij}, β_i $(i, j = 1, \ldots, n)$ satisfy the conditions

$$|\alpha_{ij}(t, x)| \leqslant k_4(\|x\|^2 + 1), \qquad |\beta_i(t, x)| \leqslant k_5(\|x\|^2 + 1)^{1/2}$$

where $k_1, \ldots, k_5$ are nonnegative constants. Under these assumptions

$$|v(t, x)| \leqslant [k_1 + k_2(T - t)], \exp[k_3(T - t), \; t_0 \leqslant t \leqslant T, \; x \in R_n$$

For the subsequent calculations we need the following formulas for the function M, whose validity follows from Condition 4:

$$M(t, S_x) = M(t, S_x^\circ) + (\nabla M(t, G_1), S_x - S_x^\circ) \tag{3.5}$$

$$M(t, S_x) = M(t, S_x^\circ) + (\nabla M(t, S_x^\circ), S_x - S_\alpha^\circ) + (N(t, G_2)(S_x - S_x^\circ), S_x - S_x^\circ) \tag{3.6}$$

$$\nabla M(t, S_x) = \nabla M(t, S_x^\circ) + N(\dot{t}, G_3)(S_x - S_x^\circ) \tag{3.7}$$

$$G_i = S_x^\circ + \theta_i(S_x - S_x^\circ), \qquad 0 < \theta_i < 1, \quad i = 1, 2, 3$$

For convenience of writing we agree to let the letter C denote different constants not depending on ε.

Theorem 1. If Conditions 1 – 5 are fulfilled, then for $t \in [t_0, T]$, $x \in R_n$ we have

$$|S(t, x) - S^\circ(t, x)| \leqslant C\varepsilon, \qquad \|S_x(t, x) - S_x^\circ(t, x)\| \leqslant C\varepsilon \tag{3.8}$$

Proof. We denote $v = S - S^\circ$. From (3.1), (3.2) and expansion (3.5) we find that $v(t, x)$ satisfies the equation

$$v_t + (Ax + \nabla M, v_x) + {}^1/_2\varepsilon\,\mathrm{Sp}(Lv_{xx}) = -{}^1/_2\varepsilon\,\mathrm{Sp}(LS_{xx}^\circ)$$

and the initial condition $v(T, x) = 0$. The lemma's hypotheses are fulfilled by virtue of Conditions 1 – 5 and, $|{}^1/_2\,\varepsilon\mathrm{Sp}(L\overset{\circ}{S}_{xx})| \leqslant C\varepsilon$. Then the first estimate in (3.8) follows at once from the lemma.

To prove the second estimate in (3.8) we differentiate (3.1), (3.2) with respect to the components of vector x and, using (3.7), we obtain, after manipulations, the system of equations and the initial conditions

$$(v_x)_t + (A' + S_{xx}N)v_x + v_{xx}[Ax + \nabla M(t, S_x^\circ)] + {}^1/_2\varepsilon[\mathrm{Sp}(Lv_{xx})]_x = -{}^1/_2\varepsilon[\mathrm{Sp}(LS_{xx}^\circ)]_x \tag{3.9}$$

$$v_x(T, x) = 0$$

We denote $v_i^*(t) = \sup|\partial v(\tau, x) / \partial x_i|$ for $\tau \in [T - t, T]$, $x \in R_n$, and, having applied the lemma to (3.9), for each of the components of vector v_x we find

$$v_i^*(t) \leqslant C_1\varepsilon + C_2(T - t)\sum_{j \neq i} v_j^*(t), \qquad i = 1, \ldots, n \tag{3.10}$$

Here C_1, C_2 are constants independent of ε. Let us subdivide the interval $[t_0, T]$ into $N \geqslant 1$ equal intervals of length d such that $Nd = T - t_n$, $(n - 1)C_2d > 1$. and let us sum the inequality (3.10). Then for $t \in [T - d, T]$ we obtain

$$\|v_x(t,x)\| \leqslant \sum_{i=1}^{n} v_i^*(T-d) \leqslant \frac{nC_1\varepsilon}{1-(n-1)C_2d} = C\varepsilon \tag{3.11}$$

Let us now consider the interval $[T-2d,\ T-d]$. Since the inequality $\|v_x(T-d,\ x)\| \leqslant C\varepsilon$ is fulfilled at the instant $T-d$, estimates of type (3.10), (3.11) hold by virtue of the lemma also for $t \in [T-2d,\ T-d]$. By arguing analogously we establish that $\|v_x(t,\ x)\| \leqslant C\varepsilon$ for all $t \in [t_0,\ T]$, $x \in R_n$. Q.E.D.

From Theorem 1 it follows that the exact solution of the Bellman equation differs from the zeroth-approximation function by a quantity of the order of ε. The following theorem is an error estimate for the first approximation.

Theorem 2. Under Conditions 1 – 5 the estimates

$$|v^1(t,\ x)| \leqslant C\varepsilon^2, \qquad \|v_x^1(t,\ x)\| \leqslant C\varepsilon^2 \tag{3.12}$$

hold for the function $v^1(t,\ x) = S - S^\circ - \varepsilon S^1$

Proof. Making use of expansion (3.6) amd of (3.1) – (3.3), we find

$$v_t^1 + (Ax + \nabla M,\ v_x^1) + {}^1/_2\, \varepsilon \mathrm{Sp}\,(Lv_{xx}^1) = -{}^1/_2\varepsilon^2 \mathrm{Sp}\,(LS_{xx}^1) - \\ - (N(S_x - S_x^\circ),\ S_x - S_x^\circ), \quad v^1(T,\ x) = 0 \tag{3.13}$$

From Conditions 4 and 5 and Theorem 1 it follows that the right-hand side in Eq.(3.13) is a quantity of the order of ε^2. Hence, by the lemma, we obtain the first estimate in (3.12). The second estimate in (3.12) is proved analogously to the second estimate in (3.8).

We now proceed to substantiate the formulas for the approximate synthesis (2.10) of the optimal control for the perturbed system. By $Z^0(t,\ x)$, $Z^1(t,\ x)$ we denote the means of the function $F(x(T))$ under the condition that at the instant t the system is in state x, while as the control in (1.1) we apply the controls of zeroth and first approximations, $u^\circ(t,\ x)$, $u^1(t,\ x)$, respectively.

Theorem 3. The estimates

$$|S(t,\ x) - Z^\circ(t,\ x)| \leqslant C\varepsilon, \qquad |S(t,\ x) - Z^1(t,\ x)| \leqslant C\varepsilon^2 \tag{3.14}$$

are valid for the functions introduced above.

Proof. The functions Z°, Z^1 are (see [4]) solutions of the following Cauchy problems:

$$Z_t^\circ + (Ax + b(t,u^\circ), Z_x^\circ) + {}^1/_2\varepsilon\, \mathrm{Sp}\,(LZ_{xx}^\circ) = 0, \quad Z^\circ(T,x) = F(x) \tag{3.15}$$
$$Z_t^1 + (Ax + b(t,u^1), Z_x^1) + {}^1/_2\varepsilon\, \mathrm{Sp}\,(LZ_{xx}^1) = 0, \quad Z^1(T,x) = F(x)$$

To shorten the writing we denote $w^\circ = Z^\circ - S^\circ$, $w^1 = Z^1 - S^\circ - \varepsilon S^1$. From relations (3.2), (3.3), (3.15), with due regard to (3.4), (2.10), we find

$$w_t^\circ + (Ax + b(t,u^\circ), w_x^\circ) + {}^1/_2\varepsilon\, \mathrm{Sp}\,(Lw_{xx}^\circ) = -{}^1/_2\varepsilon\, \mathrm{Sp}\,(LS_{xx}^\circ)$$
$$w^\circ(T,x) = 0$$
$$w_t^1 + (Ax + b(t,u^1), w_x^1) + {}^1/_2\varepsilon\, \mathrm{Sp}\,(Lw_{xx}^1) = \tag{3.16}$$
$$M(t,S_x^\circ) + \varepsilon(\nabla M(t,S_x^\circ), S_x^1) - M(t, S_x^\circ + \varepsilon S_x^1) -$$
$${}^1/_2\varepsilon\, \mathrm{Sp}\,(LS_{xx}^1), \quad w^1(T,x) = 0$$

From the properties of function $M(t,\ S_x)$ and from Conditions 2 and 5 it follows that the right-hand side of the second problem in (3.16) is a quantity of the order of ε^2. Applying the lemma to problems (3.16), we establish that

$|w^\circ(t, x)| \leqslant C\varepsilon$, $|w^1(t, x)| \leqslant C\varepsilon^2$. Hence, with the aid of Theorems 1 and 2 we obtain

$$|S(t, x) - Z^\circ(t, x)| = |S - S^\circ + S^\circ - Z^\circ| \leqslant |S - S^\circ| + |Z^\circ - S^\circ| \leqslant C\varepsilon$$
$$|S(t, x) - Z^1(t, x)| = |S - S^\circ - \varepsilon S^1 - w^1| \leqslant |S - S^\circ - \varepsilon S^1| + |w^1| \leqslant \leqslant C\varepsilon^2$$

Q.E.D.

4. Transformation of the approximate solution. Let us reduce solution (2.8) – (2.10) to a form more convenient for application. We introduce a parametric representation of the terminal surface

$$t = T = \beta_0(a), \quad x(T) = \beta(a); \quad h(\beta_0(a), \beta(a)) \equiv 0 \tag{4.1}$$

Here a is an n-dimensional vector, β_0 is a scalar function, β is a vector-valued function. These functions should satisfy the identity $h \equiv 0$ in (4.1). The vector parameter a remains constant along the optimal trajectories of the deterministic system (see (2.5), (2.6)). Then from relations (2.3), (2.6), (4.1) it follows that

$$S^\circ(t, x) = F(\beta_0(a), \beta(a)) = F[\beta_0(\psi(t, x)), \beta(\psi(t, x))] \tag{4.2}$$

From the initial condition in (2.3) it follows that on the terminal surface the derivatives of function S° can be represented as

$$S_t^\circ = F_t + \lambda h_t, \quad S_x^\circ = F_x + \lambda h_x \quad (h = 0) \tag{4.3}$$

where $\lambda = \lambda(a)$ is some function of a. Substituting equality (4.3) into Eq. (2.3), we obtain

$$F_t + \lambda h_t - H(F_x + \lambda h_x, x, t) = 0 \tag{4.4}$$

If in Eq. (4.4) we substitute H from (1.5) and replace t and x from formulas (4.1), then it turns into a nonlinear algebraic equation for determining the function $\lambda(a)$.

By $X(t)$ we denote the fundamental matrix of the homogeneous system corresponding to (1.1). This matrix is determined by the conditions

$$dX / dt = A(t)X, \quad X(t_1) = E \tag{4.5}$$

where E is the unit matrix, t_1 is a constant. As we have already noted, the vector S_x° along the optimal trajectories of the deterministic system equals, to within sign, the adjoint vector. Consequently, for a fixed a the dependency of S_x° on t is determined by the matrix $X'^{-1}(t)$. Taking further into account condition (4.3), holding at the end of the process ($t = T$), for all t, x we obtain

$$S_x^\circ = X'^{-1}(t)q(a), \quad q(a) = X'(\beta_0(a))(F_x + \lambda h_x), \quad t = \beta_0(a), \quad x = \beta(a) \tag{4.6}$$

We need to substitute the function $\lambda(a)$ into the right-hand side of the second equality in (4.6), and in the place of the arguments t, x of the functions F, h their expressions in (4.1). Relations (4.2), (4.4), (4.6) determine S°, S_x° as functions of t, a. After this, the control u° is determined from the first condition in (2.10).

In what follows we retain the former notation S°, S^1, u°, u^1 for the functions of argument t, a . Having determined the control $u^\circ(t, a)$ by the method indicated, we substitute it into system (1.1) with $\xi = 0$ and we write the solution of this system, satisfying conditions (4.1), as

$$x = X(t)\Big[X^{-1}(\beta_0(a))\beta(a) + \int_{\beta_0(a)}^{t} X^{-1}(\tau) b(\tau, u^\circ(\tau, a))\, d\tau\Big] \equiv \varphi(t, a) \qquad (4.7)$$

Relation (4.7) yields a concrete form for the function φ of (2.5), defining the field of the optimal trajectories of the deterministic problem. Solving (4.7) with respect to a, we obtain the function $\psi(t, x)$ in (2.6).

Having completed the construction of the zeroth approximation, we pass on to the first approximation. We introduce the matrix

$$\Phi(t, a) = \|\partial\varphi_i(t, a)/\partial a_j\|, \qquad i, j = 1, \ldots, n \qquad (4.8)$$

Then the solution of (2.8), (2.9) for the function S^1 and its derivatives takes, with due regard to equalities (4.6), (4.8), the form

$$S^1(t, a) = \frac{1}{2} \int_t^{\beta_0(a)} \mathrm{Sp}\left[L(\tau) X'^{-1}(\tau) \frac{\partial q(a)}{\partial a} \Phi^{-1}(\tau, a)\right] d\tau \qquad (4.9)$$

$$S_x^1 = \Phi^{-1}(t, a) S_a^1$$

Here $\partial q / \partial a$ is the matrix of the partial derivatives of the vector q from (4.6) with respect to the components of vector a. Thus, the Bellman function in the variables t, a for the zeroth and first approximations is defined by the equalities (2.1), (4.2), (4.9) and by the relations (4.1), (4.4) – (4.8) for the functions β_0, β, λ, X, q, φ, Φ. The sysnthesis of the optimal control in the zeroth and first approximations, in these same variables, is given by equalities (2.10) into which we must substitute S_x°, and S_x^1 from (4.6), (4.9).

5. Special case and examples. 1. Let us make the solution in Sects. 2, 4 concrete for the case when the function b in (1.1) and constraint (1.2) have the form

$$b(t, u) = K(t) u + c(t), \qquad \|u\| \leqslant k \qquad (5.1)$$

Here $K(t)$ is an ($n \times m$)-matrix, $c(t)$ is an n-dimensional vector-valued function, k is a positive constant bounding the vector u in absolute value. From conditions (2.10), (2.11) for the case of (5.1) follows

$$u^\circ(t, x) = -kK'(t) S_x^\circ(t, x) \| K'(t) S_x^\circ(t, x)\|^{-1}$$
$$u^1(t, x) = -kK'(t)(S_x^\circ + \varepsilon S_x^1) \| K'(S_x^\circ + \varepsilon S_x^1)\|^{-1} \qquad (5.2)$$

We accept that the process termination instant T has been fixed so that $h = T - t$. In formulas (4.1) we can set

$$\beta_0(a) = T, \qquad x(T) = \beta(a) = a \qquad (5.3)$$

i.e. as the parameter a we take the final value of the phase vector. We consider the function F in (1.6) as being independent of t, i.e. $F = F(x)$. Furthermore, we set $t_1 = T$ in relation (4.5) so that $X(T) = E$. Then equalities (4.2), (4.6), (4.7), (4.9), with due regard to (5.1) – (5.3), take the following form:

$$S^\circ = F(a), \quad S_x^\circ = X'^{-1}(t) q(a), \quad q(a) = \partial F(a) / \partial a$$

$$x = \varphi(t, a) = X(t)\left\{a + \int_T^t X^{-1}(\tau)\left[c(\tau) - \frac{kK(\tau) K'(\tau) X'^{-1}(\tau) q(a)}{\| K'(\tau) X'^{-1}(\tau) q(a)\|}\right] d\tau\right\} \qquad (5.4)$$

The approximate solution is completely determined by formulas (2.1), (5.4), (5.2), (4.8).

2. Let the assumptions in Sect. 4 be fulfilled and, besides, let the function $F(x)$ be linear in x, i.e. $F(x) = (r, x)$, where r is a constant vector. Then from relations (5.4), (5.2) ensues

$$q(a) = r, \qquad S^\circ(t, a) = (r, a), \qquad S^1 \equiv 0$$

$$u^1 = u^\circ = -kK'(t)X'^{-1}(t)\, r \, \| K'(t)\, X'^{-1}(t)\, r \|^{-1} \tag{5.5}$$

Furthermore, from (5.4) it follows that the function $\varphi(t, a)$ proves to be linear in a, so that a linear connection exists between x and a. Solution (5.5) shows that the optimal control and the functional for the perturbed problem coincide with the solution of the deterministic problem. This result is obvious: here the Cauchy problem (1.3), (1.4) admits of an exact solution $S(t, x)$, being a linear function of the phase coordinates. Therefore, the last term of Eq. (1.3), containing the second derivatives and the stipulated random perturbations, is identically equal to zero for this solution.

3. We consider an elementary system for which Eq. (1.1) is scalar and has the form

$$dx / dt = u + \xi, \qquad |u| \leqslant k, \qquad k > 0 \tag{5.6}$$

Here ξ is a white noise with a constant small intensity εg, k and g are constants, $\varepsilon \ll 1$. We are required to find a control law which minimizes the mean of the functional $F = x^2 / 2$ at a specified final instant T. The Bellman equation (1.3) and the initial condition (1.4) for system (5.6), under the assumption made, take the form

$$S_t - k|S_x| + {}^1/_2 \varepsilon g S_{xx} = 0, \qquad S(T, x) = x^2 / 2 \tag{5.7}$$

All the matrices and vectors here turn into scalars. For example (5.6) the functions entering into relations (5.1), (5.2), (5.4) take the form

$$K(t) = X(t) = 1, \quad L(t) = g, \quad c(t) = 0, \quad S_x^\circ = q(a) = a \tag{5.8}$$

Using relations (5.8), from equalities (5.4), (4.8) we obtain $x = \varphi(t, a) = a - k(t - T)\operatorname{sign} a$, $\Phi(t, a) = 1 \quad (a \neq 0)$

$$u^0(t, a) = u^1(t, a) = -k \operatorname{sign} a, \quad S^\circ(t, a) = a^2 / 2 \tag{5.9}$$

$$S^1(t, a) = \frac{1}{2} g (T - t)$$

Equalities (5.9) are valid for $a \neq 0$. The case $a = x(T) = 0$ corresponds to hitting onto the origin, which for a deterministic system can be realized, obviously, from the region $|x| \leqslant k(T - t)$. In this region the optimal control and trajectories of the deterministic system are nonunique, while the Bellman function $S^\circ = 0$ for $a = 0$. Note that for $a = 0$, $|x| \leqslant k(T - t)$, the transformation $x = \varphi(t, a)$ is not one-to-one, and in the whole region $|x| \leqslant k(T - t)$ we have the identity $a = 0$. From definition (4.8) it follows formally that $\Phi^{-1} = 0$, and then from formula (4.9) we find $S^1(t, a) = 0$ at $a = 0$. Passing to the variables t and x, we obtain, in accordance with (5.9) and with due regard to the remarks made, the approximate solution $S^\circ + \varepsilon S^1$ of problem (5.7) in the form

$$S(t, x) = {}^1/_2 [|x| + \kappa(t - T)]^2 + {}^1/_2 \varepsilon g (T - t), \quad |x| > k(T - t), \quad a \neq 0$$

$$S(t, x) = 0, \quad |x| < k(T - t), \quad a = 0 \quad (t \leqslant T) \tag{5.10}$$

The approximate solution (5.10) has discontinuities on the staight lines $|x| = k(T - t)$; in order to obtain a smooth solution we need to add to (5.10) a solution of boundary layer type close to these lines.

The example considered relates to the very few cases when we can obtain an exact solution of the Bellman equation, which we can then compare with the approximate solution (5.10). This is of interest since for the given example the hypotheses of the theorems in Sect. 3 are not fulfilled, and the question of the error estimate for solution (5.10) remains open. We remark that the solution of problem (5.7) is an even function of x, being restricted, therefore, to the region $x \geqslant 0$. Following [5], we obtain the exact solution of problem (5.7) in the form

$$S(t, x) = \int_0^\infty \left[\frac{1}{2} y^2 p(t, x-y) + \psi(y) p(t, x+y)\right] dy$$

$$p(t, x \pm y) = \frac{1}{\sqrt{2\pi\varepsilon g(T-t)}} \exp\left\{-\frac{[x-(T-t)k \pm y]^2}{2\varepsilon g(T-t)}\right\} \tag{5.11}$$

$$\psi(y) = \frac{1}{4\alpha^2}(1 - e^{-2\alpha y}) - \frac{y e^{-2\alpha y}}{2\alpha}, \quad \alpha = \frac{k}{\varepsilon g} = \text{const}, \quad x \geqslant 0$$

By means of the substitution $x = a + k(T-t)$ and of certain manipulations, solution (5.11) is brought to the form

$$S(t, x) = \frac{a^2 + \varepsilon g(T-t)}{2} + \frac{1}{\sqrt{2\pi\varepsilon g(T-t)}} \int_0^\infty \left[\left(\frac{1}{4\alpha^2} - \frac{y^2}{2}\right) - \left(\frac{1}{4\alpha^2} + \frac{y}{2\alpha}\right) e^{-2\alpha y}\right] \exp\left[-\frac{(a+y)^2}{2\varepsilon g(T-t)}\right] dy, \quad \alpha = \frac{k}{\varepsilon g} \tag{5.12}$$

$$a = x - k(T-t), \qquad x \geqslant 0$$

Let us now obtain an asymptotic representation for solution (5.12) as $\varepsilon \to 0$. To do this we evaluate the integrals occurring in (5.12) by Laplace's method [6] in turn for the various ranges of x, t. Finally, we obtain

$$S(t, x) = \frac{1}{2}[x + k(t-T)]^2 + {}^1\!/_2 \varepsilon g(T-t) + \alpha_1, \quad x > k(T-t)$$

$$S(t, x) = {}^1\!/_4 \varepsilon g(T-t) + \frac{\varepsilon^2 g^2}{8K^2} + \alpha_2, \quad x = k(T-t) \tag{5.13}$$

$$S(t, x) = \frac{\varepsilon^2 g^2}{4K^2} + \alpha_3, \quad 0 \leqslant x < k(T-t)$$

Here the quantities $\alpha_1, \alpha_2, \alpha_3$ tend to zero as $\varepsilon \to 0$ and for fixed x, t, faster than any power of ε. Taking into account the property that function S is even and comparing the approximate solution (5.10) with the asymptotic representation (5.13) of the exact solution, we convince ourselves that they coincide to within quantities of the order of ε^2 everywhere except on the straight lines $|x| = k(T-t)$, on which the approximate solution (5.10) has discontinuities. The example shows the feasibility of applying the small parameter method also for these cases when the hypotheses of the theorems in Sect. 3 are not fulfilled.

(The small parameter method for the Bellman equation, worked out above, was proposed earlier in [7]. Another approach to analogous problems is contained in the recently published paper [8]).

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ON THE THEORY OF THE SLIPPING STATE IN DYNAMICAL SYSTEMS WITH COLLISIONS

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For a wide class of systems with collisions we propose an approximate method for computing the slipping states, characterized by the same degree of completeness and labor-consumption as the known methods for computing motions of simple types. For the case when the relative acceleration of the colliding bodies varies by a linear law on the final segment of the slipping state, we have obtained an analytic expression for the state's duration factor as a function of the velocity recovery factor under impact.

Examples of the calculation of concrete models are considered. A comparison with results obtained by exact methods shows that the error does not exceed a few percents even for the first approximation. By a slipping state in a system with collisions we mean a motion accompanied on a finite time interval by an infinite sequence of instantaneous shock interactions between two fixed elements of the system. For a wide class of systems being considered the problem has been solved in [1 – 3] of determining in phase space the exact boundaries of the slipping state regions and of delineating the existence regions of periodic motions with a slipping state segment in parameter space. However, it is not advisable to recommend the use of the iterative procedure used in